Homework no. 4

1. Derive the breakup spin period of

   (a) the Sun (assume a mass of $1 M_\odot = 2 \times 10^{33}$ g and a radius of 700,000 km).

   (b) A white dwarf of 0.6 $M_\odot$ (assume a radius of 9000 km).

   (c) A neutron star of 1.4 $M_\odot$ (assume 10 km radius).

   For all three, assume a spherical body with no rotation-induced flattening.

   ANSWER: Kepler's 3rd law, applied to this situation, is:

$$V_{\rm Kep} = \sqrt{GM/R}$$

   or

$$P_{\rm bu} = 2\pi\sqrt{R^3/GM}$$

   Thus, the break-up spin period is 10074s (∼2.8 hr) for the Sun, 18.96 s for the white dwarf, and 0.46 ms for the neutron star, ignoreing rotation-induced flattening and relativistic correction and other fine details. The actual spin periods of isolated stars tend to be considerably longer than their break-up spin period.

2. Assume that the moment of inertia of neutron stars and white dwarfs can be approximated by that of a uniform sphere of the same mass and radius.

   (a) Take a 1.4 $M_\odot$, 10 km radius neutron star with a spin period of 0.1 s. What is its total rotational energy? If it also has $\dot{P} = -1.0 \times 10^{-12}\ {\rm s\,s^{-1}}$, what is the total luminosity available from this spin-down?

   (b) Reapeat the calculation for a 0.6 $M_\odot$, 9000 km radius white dwarf with a 100 s spin period and $\dot{P} = -1.0 \times 10^{-14}\ {\rm s\,s^{-1}}$.

   ANSWER: Given the assumptions, the moment of inertia of the stars are:

$$I = \frac{2}{5}MR^2$$

   In general, for a given type of star, I is proportional to MR^2 but the constant has to be evaluated from the detailed knowledge of the structure (=mass distribution) of that star. Given this, the rotational energy is:

$$E = \frac{1}{2}I\omega^2 = 2\pi^2 I/P^2$$

   and the rate of loss of rotational energy is:

$$\dot{E} = I\omega\dot{\omega} = -4\pi^2 \dot{P} P^{-3}$$

As some of you noted (or got confused by), Box 3.6 of Charles & Seward (p88) contains typos (both formulae are off by a factor of π).

The numbers for the neutron star are: rotational energy of 2.2×10^{41} J $= 2.2 \times 10^{48}$ ergs and 4.42×10^{30} W $= 4.42 \times 10^{37}$ ergs s^{-1}. For the white dwarf, they are 7.7×10^{47} ergs and 1.54×10^{32} ergs s^{-1}.

Homework no. 5

1. Mass function:

   - Cygnus X-1 has P=5.6 days and $K_2 = 76$ km s^{-1}. What is its mass function, is the units of solar masses (use $1M_\odot = 2 \times 10^{33}$ g)?
   - If M_2 is unknown, is there a solution allowed in which $M_1 < 2.0M_\odot$ for Cyg X-1?
   - The SXT XTE J1118+480 has P=0.17 days and $K_2 = 709$ km s^{-1}. What is its mass function?
   - If M_2 is unknown, is there a solution allowed in which $M_1 < 2.0M_\odot$ for XTE J1118+480?

   ANSWER: Mass function is

$$f(M) = \frac{(M_1 \sin i)^3}{(M_1 + M_2)^2} = \frac{PK_2^3}{2\pi G}$$

Substituting numerical values into the right hand side, the mass function of Cyg X-1 is 0.25 $M_\odot$. Given only this information, then there is a solution in which $M_1 < 2.0M_\odot$, such as $M_1 = 1.0$, $M_2 = 0.5$, $i = 55.6$.

The mass function for XTE J1118+480 is 6.24 $M_\odot$. Since $\sin i \le 1$ and $M_1 + M_2 \ge M_1$,

$$f(M) \le \frac{M_1^3}{(M_1 + M_2)^2} \le \frac{M_1^3}{M_1^2} = M_1$$

Thus, the mass function derived from radial velocity curve of one star is the absolute lower limit of the mass of the other star — in this case, the mass of the accreting star in XTE J1118+480 is at least 6.24 $M_\odot$, given the measured period and semi-amplitude of the secondary star.

It's mathematically possible for the accreting star in Cyg X-1 to be a neutron star (although given other bits of information, it's highly improbable), but the accreting star in XTE J1118+480 cannot be a neutron star.

Note: I've used the subscript 1 (as in M_1) and the word "primary" to refer to the accreting star throughout my lectures and problem sets. Many people, however, use the subscript 1 to refer to the more massive star. In the case of Cyg X-1, the mass-losing star is believed to be more massive, and in the case of XTE J1118+480, it's the black hole which is more massive. In this definition, the mass-losing star is the primary in Cyg X-1 and the secondary in XTE J1118+480.

2. Imagine that you have just built an adaptive optics infrared camera with an angular resolution of 0.1 arcsec (that is, a point-like source appears to have a disk of about 0.1 arcsec diameter), and that you can determine the position of bright stars accurate to 0.01 arcsec. You set out to observe the galactic center, assumed to be 8 kpc away (1 kpc = 1,000 pc; 1 pc = 3.1×10^{18} cm).

   - At the distance of the Galactic center, what linear distance does an angular separation of 0.1 arcsec correspond to?
   - If a star is in a circular Keplerian orbit around a million solar mass black hole with an orbital separation corresponding to the above distance, what would its velocity be? (Please express the answer in $\mathrm{km\,s^{-1}}$ to make it human-friendly.)
   - If a star along a straight line at the above velocity, how much (linear distance) does it move in a year? If that movement is entirely in the plane of the sky, how much (angular distance) does it move in a year?
   - Given the above simplified calculations, is your instrument useful in determining the existence or otherwise of a SMBH at the center of our Galaxy?

   ANSWER: Some of you were confused by the phrasing of this question — I apologize. The intention was "can we use an instrument like this to observe the stars near the Galactic Center black hole move on the sky? If there was a star just 0.1 arcsec from the black hole, how much could it move in a year? Can we possibly detect such a position shift?"

   The calculations only require trigonometry and Kepler's 3rd law, and the correct numbers are: 1.2×10^{14} m ($\sim$ 800 AU), 1054.5 $\mathrm{km\,s^{-1}}$, 3.33×10^{10} km, and 0.028 arcsec.

   Resolution, in this context, just refers to the size of the image of a point-like source. If you see a disk of radius 1 inch, and if someone moved it by 1/10th of an inch, you can probably see that it has moved: so the accuracy of position determination is typically many times better than the "resolution." By construction of this question, any movements of a star as small as 0.01 arcsec can be detected by this instrument. (Also, although this isn't explicitly stated, astronomers are patient — if you can see the stars move only after 5 years, such an instrument still is useful.) Thus, such an instrument is useful, and in fact helped to cement the case for a 2.6 million solar mass black hole at the center of our Galaxy.

Homework no. 6

1. The effective temperature T_{eff} of an accretion disk at radius R around a star of mass M_1 accreting at a rate of $\dot{M}$ can be written as:

$$\sigma T_{eff}^4 = \frac{3GM_1\dot{M}}{8\pi R^3}[1 - \beta(\frac{R_{in}}{R})^{1/2}]$$

R_1 is the inner radius of the accretion disk. (G: Newtonian constant of gravity = 6.673×10^{-9} cgs; σ: Stefan-Boltzmann constant = 5.67×10^{-5} cgs). In the homework handout, the parenthethes around $\frac{R_{in}}{R}$ was omitted — thus this question was not included in the grading.

- For non-rotating star (for which $\beta = 1$), derive the R at which the accretion disk is the hottest.
- If an accretion disk reaches down to the surface of a non-rotating neutron star whose mass is 1.4 $M_\odot$ ($1M_\odot = 2\times10^{33}$ g) and radius is 10 km, and the accretion rate is 10^{18} g s^{-1}, what is the maximum temperature of that disk? (Express the temperature in keV, where $1keV = 11.6\times10^6K$)?
- If the accretion disk is truncated at R=10,000 km by the magnetic field (and the above non-rotating formula still works — this assumption is only for the convenience of this question! — what is the maximum temperature of that disk?

ANSWER:

$$\frac{d}{dR}(\sigma T_{eff}^4) \propto -3R^{-4}(1-\sqrt{R_{in}/R}) - \frac{1}{2}R^{-3}(-\sqrt{R_{in}/R}) = -3R^{-4} + \frac{7}{2}R^{-4}\sqrt{R_{in}/R}$$

Thus, maximum T_{eff} is at $\frac{49}{36}R_{in} \sim 1.36R_{in}$.

Substituting in the numerical values, with R_{in}=10 km, $\sigma T_{max}^4 = 1.267\times10^{24}$ (cgs) or $kT_{max} = 1.054keV$. With R_{in} = 10,000 km, $kT_{max} = 0.0056keV$ (or $T_{max} = 65,000K$). I.e., if the accretion disk is truncated at 10,000 km, such a disk won't emit much X-rays.

2. In a magnetic cataclysmic variable system of the Intermediate Polar (IP) type, the accretion disk is truncated at R_{in} where the Keplerian period of the disk material equals the rotation period of the white dwarf. For an IP with a $1.0M_\odot$ white dwarf with a 6000 km radius, in a binary with a separation a of 1,000,000 km, derive the range of possible spin periods by considering two extremes:

- Disk extends down to the white dwarf surface.
- Disk extends down only to $0.5a$ (which, from other consideration, is actually the approximate outer radius of the disk).

In this calculation, assume that motion of particles in the disk is entirely determined by the gravity of the white dwarf.

ANSWER: This may have confused you because it's so simple. The question just asks for a straightforward application of Kepler's 3rd law, with answers 8 sec and 6081 sec, respectively. This (with minor variations due to the white dwarf mass and binary separation) is the possible range of spin periods in Intermediate Polars, the white dwarf equivalent of X-ray pulsars.